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Discrete uncertainty relations

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Abstract. Generalized uncertainty relations based upon Fourier transforms of both discrete and continuous functions are briefly reviewed. We extend these results in order to establish discrete versions of the angular momentum uncertainty relations, based upon $SU(2)$ transformations. Possible applications in quantum signal processing and computations are briefly discussed.

1. Introduction

It is well known that the familiar Heisenberg uncertainty relation for momentum p and position q, given by

$$\Delta q \Delta p \geqslant \frac{\hbar}{2} \tag{1}$$

can be viewed as a direct consequence of Fourier analysis. That is, a function and its Fourier transform cannot both be highly concentrated. Physically, this implies that the less the uncertainty in q, the greater the uncertainty in p, and conversely [1]. Here, the relation (1) is a variance-based uncertainty, whereby the expression Δq in equation (1) above is understood to be the standard deviation, defined by

$$\Delta q \equiv \sqrt{\langle\psi|(q-\bar{q}_\psi)^2|\psi\rangle} \tag{2}$$

with $\bar{q}_\psi \equiv \langle\psi|q|\psi\rangle$ for some normalized state vector $|\psi\rangle$.

Since Heisenberg's intuitive, physically motivated derivation of the uncertainty relations, a number of other approaches to such relations have been pursued. For example, in the information theoretic (or entropic) uncertainty relations, as discussed briefly below, one associates the Shannon information entropy to the measure of uncertainties. Another attractive approach is the quantum extension of the classical Cramér–Rao inequality for parameter estimation [2, 3]. This approach does not require the association of self-adjoint operators to the parameters, therefore yields results more general than the standard uncertainty relations in the sense that a large variety of physically important measurements can be treated systematically.

Although these alternative, variance-based approaches resolve some of the difficulties in obtaining various types of uncertainty relations within the conventional description of quantum mechanics, it is not obvious whether one can thereby recover operator uncertainty relations [4] such as those concerning angular momentum. Also, the assignment of uncertainty to discrete outcomes of the measurements of observables, such as spin or angular momentum, has remained an open problem up to now.

Recently, however, Donoho and Stark [5] provided, in the setting of signal processing analysis, uncertainty relations involving discrete Fourier transforms. In this paper, we first briefly outline, as a comparison, the results from entropic uncertainty relations. These results are of considerable interest in their own right since in some circumstances they are 'stronger' than the conventional Heisenberg relations. We then briefly present the results that have been obtained on generalized uncertainty relations arising in connection with Fourier analysis. These notions are then extended in order to obtain new discrete versions of angular-momentum uncertainty relation, one of which takes the form

$$N_1 N_2 \geqslant 2j + 1 \tag{3}$$

where N_i is the number of nonzero components of the angular momentum for a spin-j particle in the ith direction.

2. Entropic uncertainty relations

In quantum theory, any single observable or a commuting set of observables can in principle be measured with arbitrary accuracy. However, there is in general an irreducible lower bound on the uncertainty in the result of a simultaneous measurement of noncommuting observables. Heisenberg's relation is one such; however, Bialynicki-Birula and Mycielski [6], Deutsch [7], and others [8] argued that, in a sense, the Heisenberg inequality is 'too weak' for practical purposes, which led them to the establishment of information theoretic uncertainty relations. In the entropic uncertainty relations the information entropy

$$H(p) = -\int p(x) \ln p(x)\, \mathrm{d}x \tag{4}$$

for a given probability distribution $p(x)$, serves as an accurate measure of uncertainties. Within the context of Fourier theory, Hirschman [9] argued that if $\psi(x)$ and $\tilde{\psi}(k)$ are related through a Fourier transform

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi\hbar}} \int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx/\hbar} \psi(x) \tag{5}$$

then the following inequality holds:

$$H(|\psi|^2) + H(\hbar|\tilde{\psi}|^2) \geqslant 1 + \ln \pi. \tag{6}$$

This inequality was proved by Beckner [10] and Bialynicki-Birula and Mycielski [6], and it has been applied to a number of contexts in quantum mechanics. For example, one of Bialynicki-Birula's inequalities [11] takes the form

$$H_{\Delta x}(|\psi(x)|^2) + H_{\Delta k}(|\tilde{\psi}(k)|^2) \geqslant 1 - \ln 2 - \ln\left(\frac{\Delta x\, \Delta k}{\hbar}\right) \tag{7}$$

where $H_{\Delta x}(p(x)) = -\sum_i p_i^X \ln p_i^X$, whose left-hand side is understood to have the interpretation of [7] 'uncertainty in the result of a measurement of k and x'. Here, the notation p_i^X denotes the probability of finding the observable X in its ith interval Δx_i of the spectrum.

He also extended the argument to cover an entropic uncertainty relation for angle and angular momentum given by

$$H_{\Delta\phi}(p_\phi^m) + H_{c_m}(p_{J_z}^m) \geqslant -\ln \frac{\Delta\phi}{2\pi} \tag{8}$$

where the probabilities p_ϕ^m and $p_{J_z}^m$ are given respectively by $p_\phi^m = \int_{\Delta\phi_m} \mathrm{d}\phi\, |\psi(\phi)|^2$ and $p_{J_z}^m = |c_m|^2$, and the wavefunction

$$\psi(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} c_m \mathrm{e}^{\mathrm{i}m\phi} \tag{9}$$

depends upon the angular variable ϕ and its expansion coefficients c_m into the set of eigenfunctions of J_z. Analogous studies have also been made by Sánchez-Ruiz [12], who examined in detail the lower bound of the entropic uncertainty relation for angular momentum variables, given by

$$H(J_z) + H(J_{z'}) \geqslant -\ln\left[\binom{2j}{n^*}\left(\cos^2\frac{\beta}{2}\right)^{2j-n^*}\left(\sin^2\frac{\beta}{2}\right)^{n^*}\right] \tag{10}$$

where $n^* = [(2j+1)\sin^2(\beta/2)]$ and β is the angle between the two axis z and z'.

It is interesting to note that recently Steane [13] has extended the entropic uncertainty relation in order to obtain a discrete form of uncertainty relation, from which he made a link between basic quantum theory and the linear error correcting codes of classical information theory. He considered a 'binary' basis called 'basis 1' and another basis 'basis 2' which is obtained by rotating the original basis 1. Now, suppose a state can be written as a superposition of M_1 of the product states of basis 1 and a superposition of M_2 of the product states of basis 2. Then, Steane's inequality gives a lower bound on the product of these two numbers:

$$M_1 M_2 \geqslant 2^n \tag{11}$$

where n is the total number of 'binary states'. Note that the analysis of Steane can be extended to the context of angular momentum studied in [12]. Specifically, we obtain†

$$N_z N_{z'} \geqslant \left\{\binom{2j}{n^*}\left(\cos^2\frac{\beta}{2}\right)^{2j-n^*}\left(\sin^2\frac{\beta}{2}\right)^{n^*}\right\}^{-1}. \tag{12}$$

For further details on entropic inequalities, we refer to the above-mentioned references. Here, instead, we shall first review some recent developments in signal processing, then extend these notions in order to obtain new discrete versions of the angular momentum uncertainty relations.

3. Fourier-based inequalities

Let us start by providing a number of standard definitions and theorems. The functions and sequences used are all elements of L_2 or l_2, with unit norm, unless otherwise specified. The discrete Fourier transform of a sequence $\{x_t\}$ of length N is defined as

$$\tilde{x}_\omega \equiv \frac{1}{\sqrt{N}}\sum_{t=0}^{N-1} x_t \mathrm{e}^{-2\pi \mathrm{i}\omega t/N} \tag{13}$$

and for the continuous Fourier transform, we have

$$\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\mathrm{e}^{-2\pi\mathrm{i}\omega t}. \tag{14}$$

Now, we introduce two operators $\hat{P}_T$ and $\hat{P}_\Omega$. The time limiting operator $\hat{P}_T$ is defined as

$$(\hat{P}_T f)(t) = \begin{cases} f(t) & t \in T \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

and the frequency limiting operator $\hat{P}_\Omega$ is

$$(\hat{P}_\Omega f)(t) = \int_\Omega \mathrm{d}\omega\, \mathrm{e}^{2\pi\mathrm{i}\omega t}\tilde{f}(\omega) \tag{16}$$

† This result was pointed out to us by the anonymous referee of *J. Phys. A: Math. Gen.*

where T and Ω are arbitrary measurable subsets on the real line. The norm of a bounded operator Q is defined to be

$$\|Q\| = \sup_{g\in L_2} \frac{\|Qg\|}{\|g\|} \tag{17}$$

with the L_2-norm of a function f given by $\|f\|^2 = \int_{-\infty}^{\infty} \mathrm{d}t\, |f(t)|^2$. It follows from Parseval's identity that the L^2-norms of a function f and its Fourier transform $\tilde{f}$ agree: $\|f\| = \|\tilde{f}\|$.

We say that a function f is ϵ_T-concentrated if $\|f - \hat{P}_T f\| \leqslant \epsilon_T$ for a real number ϵ_T and similarly $\tilde{f}$ is said to be ϵ_Ω-concentrated if $\|f - \hat{P}_\Omega f\| \leqslant \epsilon_\Omega$. Hence if f is ϵ_T-concentrated on a measurable set T and $\tilde{f}$ is ϵ_Ω-concentrated on a measurable set Ω, we have

$$\|f - \hat{P}_\Omega \hat{P}_T f\| \leqslant \epsilon_T + \epsilon_\Omega. \tag{18}$$

Next, defining the operator Q by $(Qf)(t) \equiv \int_{-\infty}^{\infty} q(s,t) f(s)\, \mathrm{d}s$, the Hilbert–Schmidt norm of Q is then given by

$$\|Q\|_{HS} \equiv \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{d}s\, |q(s,t)|^2 \right)^{1/2}. \tag{19}$$

Note that the two norms are related by $\|Q\| \leqslant \|Q\|_{HS}$.

With these definitions at hand, we can verify that the Hilbert–Schmidt norm of $\hat{P}_\Omega \hat{P}_T$ is given by $\|\hat{P}_\Omega \hat{P}_T\|^2_{HS} = |T||\Omega|$.

The proof is sketched as follows. First, we write the Hilbert–Schmidt norm of $\hat{P}_\Omega \hat{P}_T$ as

$$\begin{aligned}(\hat{P}_\Omega \hat{P}_T f)(s) &= \int_\Omega \mathrm{d}\omega\, \mathrm{e}^{2\pi \mathrm{i}\omega s} \int_T \mathrm{d}t\, \mathrm{e}^{-2\pi \mathrm{i}\omega t} f(t) \\ &= \int_T \left(\int_\Omega \mathrm{e}^{2\pi \mathrm{i}\omega(s-t)} \right) f(t)\end{aligned} \tag{20}$$

so that $(\hat{P}_\Omega \hat{P}_T f)(s) = \int_{-\infty}^{\infty} \mathrm{d}t\, q(s,t) f(t)$, where $q(s,t)$ is defined as

$$q(s,t) = \begin{cases} \int_\Omega \mathrm{d}\omega\, \mathrm{e}^{2\pi \mathrm{i}\omega(s-t)} & t \in T \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Then, the Hilbert–Schmidt norm can be calculated to be

$$\begin{aligned}\|\hat{P}_\Omega \hat{P}_T\|^2_{HS} &= \int_T \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}s\, |q(s,t)|^2 \\ &= \int_T \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}\omega\, |1_\Omega \mathrm{e}^{-2\pi \mathrm{i}\omega t}|^2 \\ &= \int_T \mathrm{d}t\, |\Omega| = |T||\Omega|\end{aligned} \tag{22}$$

where we have used Parseval's identity, and 1_Ω denotes the indicator function of the set Ω.

Using the relation introduced above we are now in a position to note a number of generalized uncertainty relations. First, we prove the following lemma for discrete functions.

Lemma (Donoho and Stark). *If the sequence $\{x_t\}$ of length N has N_t nonzero elements, then $\{\tilde{x}_\omega\}$ cannot have N_t consecutive zeros.*

Let $\{y_\tau\}$ $(\tau = 1, \ldots, N_t)$ denote the nonzero elements in $\{x_t\}$. Then, if we denote the frequency interval under consideration by $\omega = l+1, \ldots, l+N_t$ for some $l \in (0, N-1)$, we can write

$$g_k \equiv \tilde{x}_{l+k} = \frac{1}{\sqrt{N}} \sum_{\tau=1}^{N_t} y_\tau (\mathrm{z}_\tau)^{l+k} \tag{23}$$

where $z_\tau = \exp[-2\pi i/N \cdot \tau]$. Defining the matrix $Z_{k\tau} = (z_\tau)^{l+k}/\sqrt{N}$, the above relation (23) can then be rewritten in vector notation as $\boldsymbol{g} = \boldsymbol{Z}\boldsymbol{y}$. Since $\boldsymbol{y} \neq \boldsymbol{0}$ by definition, if $\boldsymbol{g} = \boldsymbol{0}$, then the matrix $\boldsymbol{Z}$ must be singular. However, by definition, $\boldsymbol{Z}$ is an $N_t \times N_t$ Vandermonde matrix which is known to be nonsingular [14], and hence $\boldsymbol{g} \neq \boldsymbol{0}$, i.e., $\{\tilde{x}_\omega\}$ cannot have N_t consecutive zeros. A consequence of this lemma is the following theorem.

Theorem (Donoho and Stark). *Suppose that $\{x_t\}$ is nonzero at N_t points and that $\{\tilde{x}_\omega\}$ is nonzero at N_ω points. Then, the following inequalities hold:*

$$N_t \cdot N_\omega \geqslant N \tag{24}$$

$$N_t + N_\omega \geqslant 2\sqrt{N}. \tag{25}$$

The proof for the first inequality follows directly from the above lemma, since a sequence without gaps always has the correct number of nonzero elements to satisfy the inequality. The second inequality follows immediately from the first, since for all positive numbers the arithmetic mean is larger than the geometric mean.

A useful theorem for the continuous functions can be obtained by combining equation (18) and the Hilbert–Schmidt bound for the operator norm, and by use of the triangle inequality.

Theorem (Donoho and Stark). *Let T and Ω be measurable sets on the real line, and suppose there is a Fourier transform pair $(f, \tilde{f})$ such that f is ϵ_T-concentrated on T and $\tilde{f}$ is ϵ_Ω-concentrated on Ω. Then,*

$$|\Omega||T| \geqslant (1 - (\epsilon_T + \epsilon_\Omega))^2. \tag{26}$$

An analogous result can be obtained for the discrete cases, which will be discussed later. We note that, in the case where the sets T and Ω are both single intervals, a sharper bound was obtained by Slepian, Landau, and Pollak [16, 17]. They have shown that, in this case, the norm $\|\hat{P}_\Omega \hat{P}_T\|$ is the largest eigenvalue of the operator $\hat{P}_\Omega \hat{P}_T \hat{P}_\Omega$, and the eigenvalue analysis of this operator was studied.

4. Angular-momentum uncertainty relations

In this section we demonstrate that the discrete uncertainty relations above can be generalized to provide discrete analogues of the standard angular momentum uncertainty relations, for example,

$$(\Delta \hat{J}_x)^2 (\Delta \hat{J}_y)^2 \geqslant \frac{\hbar^2}{4} \langle \hat{J}_z \rangle^2. \tag{27}$$

Note that the average $\langle \cdot \rangle$ depends upon the state of the system. Hence, if one chooses an eigenstate of $\hat{J}_z$, for example, then the right-hand side is just $m^2\hbar^4/4$.

As an illustration, consider now the situation where one observes an ensemble of polarized spin particles with unknown polarization direction, but knows that all the particles are identically prepared. These particles may be regarded, for example, as hydrogen atoms or alkaline-earth metal atoms which do not exhibit the anomalous Zeeman effect. For simplicity, we assume that the observer uses Stern–Gerlach devices with magnetic fields directed at angles θ_1 or θ_2 relative to the x-axis. The detectors are then screens along the θ_1- and θ_2-axes. Each screen is divided into $2j + 1$ intervals (i.e., one-dimensional boxes) where j is the highest spin of the atom, which is assumed known. After observations of the ensemble (where the number of particles is assumed to be $\gg 2j + 1$) along, say, the direction of the θ_2-axis, one obtains an assignment of numbers to the respective boxes.

Suppose that, as a result of the observations, one obtains a distribution sharply peaked somewhere along the θ_2-axis. The commutation relations between angular momentum operators imply the impossibility of simultaneously determining more than one component of the angular momentum. Hence, if we measure the θ_1-component of the same atoms, we intuitively expect to obtain a widely spread distribution. If these (θ_1 and θ_2) distributions were related by a Fourier transform, then from the above arguments we would obtain

$$N_1 \cdot N_2 \geqslant 2j + 1 \tag{28}$$

where N_i $(i = 1, 2)$ denotes the number of nonzero elements (nonempty boxes) along the θ_i-axis. However, the spinor components with squared amplitudes defining these two distributions are related by the following rotation matrix [15]:

$$d^{(j)}_{m'm}(\beta) = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{\frac{1}{2}} \left(\cos\frac{\beta}{2}\right)^{m'+m} \left(\sin\frac{\beta}{2}\right)^{m'-m} P^{m'-m,m'+m}_{j-m'}(\cos\beta) \tag{29}$$

rather than by a Fourier transform. Here, P is the Jacobi polynomial

$$P^{a,b}_n(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-a}(1+x)^{-b}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-x)^{a+n}(1+x)^{b+n}] \tag{30}$$

and the angle β in our case is $\theta_2 - \theta_1$. Hence, it is not clear that an inequality such as (28) should hold. However, in the following we shall prove that all the relevant submatrices of $d^{(j)}_{m'm}$ have nonvanishing determinants except for few values of β belonging to a set of measure zero, and hence, following the same argument as in the Fourier transform case, the above relation (28) is indeed valid.

In order to prove this, we must first show that, for any fixed value of m' (that is, a fixed row of the matrix $d^{(j)}_{m'm}$), any $j + 1 + m'$ $(m' = -j, \ldots, +j)$ elements of the row are independent functions. That is, all the $2j + 1$ elements in the first row of the matrix are independent, any $2j$ elements of the second row are independent, any $2j - 1$ elements of the third are independent, and so on. If this is the case, which we shall prove below, then all the relevant submatrices are Wronskian matrices [14] of independent functions which are known to have nonzero determinant. This implies that the distribution of the numbers along the θ_1-axis cannot have N_2 consecutive zeros, and hence (28) follows.

Since we are interested in the determinants of the submatrices under consideration, we omit some irrelevant parts, that is, parts common to the rows or to the columns, in expressions (29) and (30). Performing a simple coordinate transformation from x to $z = 1 + x$, we obtain

$$d^{(j)}_{m'm} \sim \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[\sum_k \binom{j-m}{k}(-1)^{j-m-k}2^k z^{2j-k}\right] \to {\sum_k}' \binom{j-m}{k} z^{j+m'-k} \tag{31}$$

where $\sum'$ denotes a summation with the restrictions $j + m' \geqslant k$ and $j - m \geqslant k$, and we have also omitted some irrelevant factorials. Now, for any fixed row m', we denote the indices m of the arbitrary chosen columns in this row by m_i, where $i = 1, \ldots, j + 1 + m'$. This label i corresponds to the label τ previously used in describing the Fourier transform. We then expand this row into a matrix, denoted by B_{ik}, by varying the label k, i.e.,

$$B_{ik} = \begin{cases} \binom{r_i}{k} & r_i \equiv j - m_i \geqslant k \\ 0 & \text{otherwise.} \end{cases}$$

If we define $\hat{B}_{ik} = r_i(r_i - 1)\ldots(r_i - k + 1)z^{r_i}$, then $\hat{B}$ is nonsingular if and only if B is. However, $\hat{B}$ can be viewed as a Wronskian matrix for the monomial of power r_i. Moreover, $\hat{B}$ is nonsingular since $r_i \neq r_j$ for $i \neq j$. Thus, we have shown that any $j + 1 + m'$ elements in row m' are independent, as required.

The next step is to prove that the submatrix defined by an arbitrary set of N_2 number of columns and the same number of consecutive rows (e.g. $m' = l+1, \ldots, l+N_2$) is a Wronskian matrix of independent polynomials. However, this follows directly from (31). It is now clear by the same arguments as those used above for the case of Fourier transforms that the spinor components relative to the θ_1-axis cannot have N_2 consecutive zeros. However, we are merely interested in the nonzero elements of the probability distributions, which simply correspond to the nonzero spinor components. Hence, we are led to the following result:

Theorem. *Let N_1 be the number of nonzero spin-components of a spin-j particle in any given one direction, and let N_2 be the number of nonzero spin-components in a different direction. Then, apart from a number of directions belonging to a set of measure zero, the following inequality holds:*

$$N_1 \cdot N_2 \geqslant 2j + 1. \tag{32}$$

As a trivial extension, we can obtain the following bound for simultaneous measurements of k different components of the angular momentum of the atoms.

Corollary. *Let $\{N_k\}$ be the set of numbers of nonzero spin-components in k distinct directions. Then,*

$$N_1 \cdot N_2 \ldots N_k \geqslant (2j + 1)^{k/2}. \tag{33}$$

Note that, although the above proof follows for almost any angle β, in order that the inequality be valid for the case $\beta \ll 1$, the required sample size must approach infinity. Also, as mentioned above, the set of angles where the determinants of some submatrices vanish is of measure zero. For example, it can be shown that the inequality (28) does not hold for any nonzero spin j if the angle $\beta = n\pi$.

5. Bandlimited uncertainty relations

In the foregoing discussion, we have assumed an ideal situation where the number of particles is large, and the observer can ascertain with sufficient certainty whether or not any particles have arrived at any given box. Otherwise, we can block some of the detector boxes and confine our attention to the particles passing through the remaining boxes. After these particles have passed through the field of the Stern–Gerlach magnet in the first direction, we recombine the beams and then measure the components in another direction. For such cases, as well as in general, an approximate version of the above inequality is useful. We shall consider this for both Fourier transforms and spinor rotations in the following discussion.

Again, we consider two distributions f_m and $\tilde{f}_{m'} = \sum_m d_{m'm} f_m$. If $d_{m'm}$ is an element of $U(1)$, then this is just a discrete Fourier transform, and if $SU(2)$, then f_m denote the normalized spinor components corresponding to one axis and $\tilde{f}_{m'}$ be those corresponding to another axis, rotated by the angle β relative to one. We now define two projection operators, given by

$$\hat{P}_T f_n = \begin{cases} f_n & n \in T \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

and

$$\hat{P}_\Omega f_n = \sum_{m \in \Omega} d^\dagger_{nm} \tilde{f}_m \tag{35}$$

for some index sets T and Ω. The sequence f_n is said to be ϵ_T-concentrated on an index set T if $\|f - \hat{P}_T f\| = \sum_n |f_n - \hat{P}_T f_n| \leqslant \epsilon_T$, and similarly, $\tilde{f}$ is ϵ_Ω-concentrated on Ω if

$\| f - \hat{P}_\Omega f \| \leqslant \epsilon_\Omega$. Therefore, if f is ϵ_T-concentrated on T and $\tilde{f}$ is ϵ_Ω-concentrated on Ω, we have $\| f - \hat{P}_\Omega \hat{P}_T f \| \leqslant \epsilon_T + \epsilon_\Omega$, whence it follows that $\| \hat{P}_\Omega \hat{P}_T \| \geqslant 1 - \epsilon_T - \epsilon_\Omega$. We now find an upper bound for the norm of $\hat{P}_\Omega \hat{P}_T$. Consider first Fourier transforms. In this case, the bound can simply be obtained by introducing the Frobenius matrix norm $\| \hat{Q} \|_F^2 \equiv \sum_{m,k} q_{mk}^* q_{mk}$ of an operator $\hat{Q}$, which is defined by $\hat{Q} f_m \equiv \sum_n q_{mn} f_n$.

Since the conventional norm of an operator satisfies the inequality $\| \hat{Q} \| \leqslant \| \hat{Q} \|_F$ [5], we now calculate the norm $\| \hat{P}_T \hat{P}_\Omega \|_F$. In terms of the matrix elements, one obtains

$$\hat{P}_\Omega \hat{P}_T f_k = \sum_m q_{km} f_m \tag{36}$$

with

$$q_{km} = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{m' \in \Omega} \mathrm{e}^{2\pi \mathrm{i} m'(k-m)/N} & m \in T \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Using the Parseval's equality (i.e., the norm of an operator is the same as that of its Fourier transform), the Frobenius matrix norm can easily be calculated as $\| \hat{P}_\Omega \hat{P}_T \|_F^2 = N_T \cdot N_\Omega / N$. Hence, we are led to the following result.

Theorem (Donoho and Stark). *Let $\{(f_t), (\tilde{f}_\omega)\}$ be a Fourier transform pair with (f_t) ϵ_T-concentrated on the index set T and $(\tilde{f}_\omega)$ ϵ_Ω-concentrated on the index set Ω. Let N_T and N_Ω denote the number of elements of T and Ω, respectively. Then,*

$$N_T \cdot N_\Omega \geqslant N(1 - (\epsilon_T + \epsilon_\Omega))^2. \tag{38}$$

Now, suppose we have a measurement that is restricted to the set T containing N_T elements (boxes), from which ϵ_T can be evaluated. After choosing N_Ω boxes for the second measurement, we find an upper bound for the observed intensity $1 - \epsilon_\Omega$ given by

$$1 - \epsilon_\Omega \leqslant \sqrt{\frac{N_T N_\Omega}{N}} + \epsilon_T \tag{39}$$

assuming $1 - (\epsilon_T + \epsilon_\Omega) \geqslant 0$.

Note that the above results (26) and (38) can be improved further by exploiting the fact that q_{km} is a positive matrix, hence the Perron–Frobenius theorem can be applied to study the largest eigenvalue of the matrix, with the result [18]

$$S\left(\frac{N_T \cdot N_\Omega}{N}\right) \geqslant (1 - (\epsilon_T + \epsilon_\Omega))^2 \tag{40}$$

where $S(x)$ is defined as $S(x) \equiv \frac{2}{\pi}\mathrm{Si}(x) - \frac{1}{\pi}\sin(x)$, with the sine integral $\mathrm{Si}(x)$ which admits the power series expansion

$$\mathrm{Si}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!(2n+1)}. \tag{41}$$

Returning to the case of angular momentum, in terms of the matrix elements $d_{m'm}$ given in (29), one obtains

$$\begin{aligned} \hat{P}_\Omega \hat{P}_T f_k &= \sum_{m' \in \Omega} d_{km'}^\dagger \sum_{m \in T} d_{m'm} f_m \\ &= \sum_{m \in T} \left(\sum_{m' \in \Omega} d_{km'}^\dagger d_{m'm} \right) f_m \\ &= \sum_m q_{km} f_m \end{aligned}$$

with

$$q_{km} = \begin{cases} \sum\limits_{m' \in \Omega} d^{\dagger}_{km'} d_{m'm} & m \in T \\ 0 & \text{otherwise.} \end{cases}$$

The resulting bound obtained by the Frobenius matrix norm in this case is not sharp and thus one must look into the possibility of using different methods. In particular, it is clear that the norm depends not only upon N_T and N_Ω but also upon the turning angle β. Moreover, if $\beta \ll 1$, then it is always possible to find an example where the above (38) does not hold. Therefore, we restrict our attention to the case $\beta \sim \pi/2$. In this case, the bound can easily be obtained, by finding the largest element of the matrix, with the result

$$\binom{N}{N/2} N_T N_\Omega \left(\sin\frac{\beta}{2} \cos\frac{\beta}{2} \right)^N \geqslant (1 - (\epsilon_T + \epsilon_\Omega))^2. \tag{42}$$

6. Discussion

We have generalized the uncertainty relations that have previously been developed for Fourier transforms to $SU(2)$ transformations in order to derive discrete angular momentum uncertainty relations. However, there are many possible applications within the framework of Fourier transforms, including position–momentum or time–energy uncertainty relations. While conventional uncertainty relations assume that the concentrations of the function (distribution) and its Fourier transform are on continuous intervals, the above results (24) and (25) are valid for concentrations on arbitrary measurable sets. Moreover, as shown in [5] for the case of Fourier transforms, the inequality (38) also holds for continuous functions. These inequalities also have important applications to signal recovery in quantum communication theory and quantum spectroscopy.

Another obvious application would be to discrete quantum mechanics [19], where generalized uncertainty relations for Fourier theory discussed above can be applied without any modification of the proofs. Also, these generalizations can be further extended to obtain uncertainty relations on Lie groups, with obvious applications in the field of particle physics, including measurements of discrete quantities such as isospin or hypercharge, etc. Such extensions can be studied by using the Peter–Weyl theorem [20].

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